MASSEY PRODUCTS AND CRITICAL POINTS

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Abstract. In this paper we use cup-products and higher Massey products to find topological lower bounds on the minimal number of geometrically distinct critical points of any closed 1-form in a given cohomology class.

§1. Introduction

Let X be a closed manifold and let $\xi \in H^1(X;\mathbf{R})$ be a nonzero cohomology class. The well-known Novikov inequalities [N1, N2] estimate the numbers of critical points of different indices of any closed 1-form ω on X lying in the class ξ, assuming that all the singular points are non-degenerate in the sense of Morse. Novikov type inequalities were generalized in [BF1] for closed 1-forms with more general singularities (non-degenerate in the sense of Bott). In [BF2] an equivariant generalization of the Novikov inequalities was developed.

Novikov inequalities have found important applications in symplectic topology, especially in the study of symplectic fixed points (Arnold's conjecture). Here we should mention the work of J.-C. Sikorav [S], Hofer - Salamon [HS], Van - Ono [VO], and most recently the preprint of Eliashberg and Gromov [EG].

In this paper we announce a new theorem, which provide topological restrictions on the number of geometrically distinct critical points of closed 1-forms. We impose no assumptions on the nature of the critical points. Therefore, the result of this paper has the same relation to the classical Lusternik - Schnirelman - Frolov - Elsgoltz theory, as Novikov's theory has to the classical Morse theory.

On any closed n-dimensional manifold, in any integral cohomology class ξ, there always exists a closed 1-form with $\leq n-1$ critical points (cf. Theorem 4.1). Our Main Theorem 2.3 proves that on some manifolds any closed 1-form has at least $n-1$ critical points.

The main technical tool of our proof is *the deformation complex*, cf. [F3], [F4]. It has advantages compared with the "Novikov's complex" (which uses formal power series).

Different estimates on the number of critical points of closed 1-forms were recently suggested in [F2, F3], where we used flat line bundles described by complex numbers, which are not Dirichlet units. There are examples, when the approach of this paper gives stronger estimates than the approach of [F2, F3], although in some other cases the situation is the opposite.

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§2. Main Theorem

2.1. Massey products. We will deal here with a special kind of higher Massey operations d_r (where $r = 1, 2, \dots$), determined by a one-dimensional cohomology class $\xi \in H^1(X; \mathbf{Z})$. The first operation $d_1 : H^i(X; \mathbf{k}) \to H^{i+1}(X; \mathbf{k})$ is the usual cup-product

$$d_1(v) = (-1)^{i+1} v \cup \xi, \qquad \text{for} \quad v \in H^i(X; \mathbf{k}). \tag{1}$$

The higher Massey products $d_r : E_r^* \to E_r^{*+1}$ with $r \geq 2$, are defined as the differentials of a spectral sequence (E_r^*, d_r), $(r \geq 1)$ with the initial term $E_1^* = H^*(X; \mathbf{k})$ and the initial differential $d_1 : E_1^* \to E_1^{*+1}$ given by (1). Each subsequent term E_r^* is the cohomology of the previous differential d_{r-1}:

$$E_r^* = \ker(d_{r-1}) / \operatorname{im}(d_{r-1}). \tag{2}$$

Traditionally, the following notation is used

$$d_r(v) = (-1)^{i+1} \cdot \langle v, \underbrace{\xi, \xi, \dots, \xi}_{(r \text{ times})} \rangle, \qquad v \in H^i(X; \mathbf{k}). \tag{3}$$

This spectral sequence was first used in the context of the Novikov inequalities in paper [F], page 46, where we proved that the Novikov - Betti number $b_i(\xi)$ coincides with $\dim E_\infty^i$. In [N3] and in [P] it is shown that the differentials d_r are given by the Massey products (3).

We describe this spectral sequence in full detail in [F4], cf. also [F], [N3], [P], [F1].

Definition. A cohomology class $v \in H^i(X; \mathbf{k})$ is said to be ξ-*surviver* if $d_r(v) = 0$ for all $r \geq 1$.

2.2. Notation. Let $\mathbf{k}$ be a fixed algebraically closed field. The most important cases, which the reader should keep in mind are $\mathbf{k} = \mathbf{C}$ or $\mathbf{k}$ being the algebraic closure of a finite field $\mathbb{F}_p$.

We will consider flat $\mathbf{k}$-vector bundles E over a compact polyhedron X. We will understand such bundles as locally trivial sheaves of $\mathbf{k}$-vector spaces. The cohomology $H^q(X; E)$ will be understood as the sheaf cohomology.

A flat vector bundle is determined by its monodromy – linear representation of the fundamental group $\pi_1(X, x_0)$ on the fiber E_0 over the base point x_0, which is given by the parallel transport along loops. For example, a flat $\mathbf{k}$-line bundle is determined by a homomorphism $H_1(X; \mathbf{Z}) \to \mathbf{k}^*$; here $\mathbf{k}^*$ is considered as a multiplicative abelian group.

The following Theorem is the main result of this paper.

2.3. Theorem. *Let X be a closed manifold and $\xi \in H^1(X; \mathbf{Z})$ be an integral cohomology class. Suppose that there exists a nontrivial cup-product*

$$v_1 \cup v_2 \cup \cdots \cup v_m \neq 0, \tag{4}$$

where the first two classes $v_1 \in H^{d_1}(X; \mathbf{k})$ and $v_2 \in H^{d_2}(X; \mathbf{k})$ are ξ-survivors and for $i = 3, \dots, m$ the classes $v_i \in H^{d_i}(X; E_i)$ belong to the cohomology of some flat

$\mathbf{k}$-vector bundles E_i over X with $d_i > 0$. Then for any closed 1-form ω on X lying in class ξ,

$$\mathrm{cat}(S(\omega)) \geq m - 1 \tag{5}$$

where $S(\omega)$ denotes the set of critical points of ω and cat *denotes the Lusternik - Schnirelman category. In particular, the total number of geometrically distinct critical points satisfies*

$$\#S(\omega) \geq m - 1. \tag{6}$$

Recall that a point $p \in X$ is a critical point of ω if $\omega_p = 0$.

In case $\xi = 0$ (when we study critical points of functions) the class $1 \in H^0(X; \mathbf{k})$ is a ξ-surviver. Hence in this case we may take $v_1 = v_2 = 1$. This shows that in the case of functions Theorem 2.3 is reduced to the usual Lusternik - Schnirelman inequality: $\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(X) + 1$, cf. [DNF].

§3. Examples

3.1. Detecting ξ-survivors. The following criterion allows us to show in some cases that a given cohomology class $v \in H^i(X; \mathbf{k})$ is a ξ-surviver, where $\xi \in H^1(X; \mathbf{Z})$.

Suppose that we may realize ξ by a smooth codimension one submanifold $V \subset X$, having a trivial normal bundle, and we may realize the class v by a simplicial cochain c, such that the support of c is disjoint from V. Then v is a ξ-surviver.

Indeed, this follows immediately from the definition of Massey operations.

This applies to the class ξ itself and shows that it is a ξ-surviver, since we may realize it by a cochain with support on a parallel copy of V. However, this observation is useless in producing nontrivial products as in Theorem 2.3, because of the presence of another ξ-surviver, which kills the whole product.

3.2. Example. Let $X = \mathbf{RP}^n \# (S^1 \times S^{n-1})$ (the real projective space with a handle of index 1), and let $\xi \in H^1(X; \mathbf{Z})$ be a class which restricts as the generator along the handle and which is trivial on the projective space. We may realize this class ξ by a sphere $V = S^{n-1}$ cutting the handle. Let $\mathbf{k} =$ be the algebraic closure of $\mathbf{Z}_2$. We have a class $v \in H^1(X; \mathbf{k})$, which restricts as an obvious generator of $H^1(\mathbf{RP}^n; \mathbf{k})$ and which is trivial on the handle. Applying the above criterion, we see that v is a ξ-surviver (since we may realize it by a chain with support on the projective space, i.e. disjoint from V). We obtain $v^n \neq 0$ and hence by Theorem 2.3 and closed 1-form ω in class ξ has at least $n - 1$ geometrically distinct critical points.

3.3. Example. Let Σ_g be a Riemann surface of genus $g > 1$. Consider the classes $v_1, v_2, \xi \in H^1(\Sigma_g; \mathbf{Z})$ which are Poincarè dual to the curves shown in Figure 1.

Then by 3.1, v_1 and v_2 are ξ-survivors. Since $v_1 \cup v_2 \neq 0$, we obtain that any closed 1-form has at least 1 critical point. This however is a trivial corollary of the Hopf's theorem.

Let now Y be an arbitrary closed manifold and let X be $\Sigma_g \times Y$. Let $\xi' \in H^1(X; \mathbf{Z})$ denote the class with $\xi'|_{\Sigma_g} = \xi$ and $\xi'|_Y = 0$.

We will show that:

any closed 1-form on X lying in class ξ' has at least $\mathrm{cl}_{\mathbf{k}}(Y)$ *critical points, where* $\mathrm{cl}_{\mathbf{k}}(Y)$ *is the cohomological cup-length of $H^*(Y; \mathbf{k})$.*

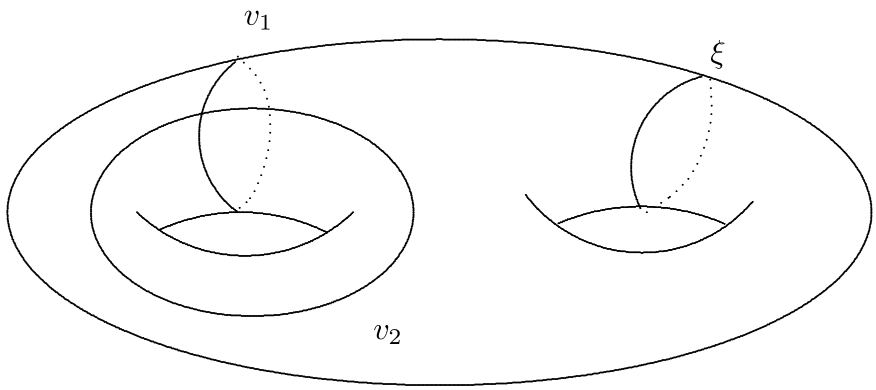

Figure 1

Indeed, suppose that $u_j \in H^{d_j}(Y;\mathbf{k})$ are some classes $j = 1, 2, \dots, r = \mathrm{cl}_{\mathbf{k}}(Y)$ such that $d_j > 0$ and $u_1 \cup u_2 \cup \cdots \cup u_r \neq 0$. Then we have $r+2$ cohomology classes

$$v_1 \times 1,\ v_2 \times 1,\ 1 \times u_1, \dots, 1 \times u_r \in H^*(X;\mathbf{k})$$

and their cup-product is nontrivial. Since the first two classes $v_1 \times 1, v_2 \times 1$ are ξ'-survivors (as easily follows from 3.1), we obtain our italicized statement using Theorem 2.3.

This construction produces many examples with large numbers of critical points.

For instance, one may take Y being the real projective space $\mathbf{RP}^{n-2}$ and $\mathbf{k}$ being the algebraic closure of the field $\mathbf{Z}_2$. Then we obtain an n-dimensional manifold

$$X = \Sigma_g \times \mathbf{RP}^{n-2}, \qquad g > 1,$$

and the above arguments prove that *any closed 1-form on X has at least $n-1 = \dim X - 1$ geometrically distinct critical points.*

3.4. Formal spaces. If X is a *formal space* [DGMS] then all higher Massey products vanish. We obtain in such a situation that *a class $v \in H^i(X;\mathbf{Q})$ is a ξ-surviver if and only if $v \cup \xi = 0$.* In this case the statement of Theorem 2.3 simplifies.

According to [DGMS], the vanishing of all higher Massey products takes place in any compact complex manifold X for which the *dd^c-Lemma* holds (for example, if X is a Kähler or a Moišezon space). This vanishing of higher products directly follows from the diagram

$$\{\mathcal{E}^*_M, d\} \xleftarrow{i} \{\mathcal{E}^{\mathfrak{c}}_M, d\} \xrightarrow{\rho} \{H_{d^c}(M), d\}$$

used in the first proof of the Main Theorem [DGMS], cf. page 270.

3.5. Remark. *Theorem 2.3 becomes false if we allow products (4) with only one ξ-surviver instead of two.*

Indeed, let $X = S^1 \times Y$ and let $\xi \in H^1(X;\mathbf{Z})$ be the cohomology class of the projection $X \to S^1$. Then ξ can be realized by a closed 1-form without critical points (the projection). However, products of the form $\xi \cup v_1 \cup \cdots \cup v_m$, where v_i are pullbacks of some classes of Y, may be nontrivial for $m = \mathrm{cl}_{\mathbf{k}}(Y)$. Note that these cup-products have only one ξ-surviver.

§4. Colliding critical points

The following result shows that the examples described in §3 are best possible.

4.1. Theorem. *Let X be a closed connected n-dimensional manifold, and let $\xi \in H^1(X;\mathbf{Z})$ be a nonzero cohomology class. Then there exists a closed 1-form ω on X, realizing ξ, having at most $n-1$ critical points.*

A proof can be obtained easily, using the method of Takens [T]. It consists in colliding the Morse critical points of the same dimension into one degenerate critical point.

Details and proofs will appear in [F4].

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